

The University of Chicago

QUINTUPLES OF
THREE-DIMENSIONAL VARIETIES
IN A FOUR-DIMENSIONAL
LINEAR SPACE

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1930

By
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I. INTRODUCTION

In a paper published in the Transactions of the American Mathematical Society, 1928, entitled "The Projective Differential Geometry of Systems of Linear Homogeneous Differential Equations of the First Order", Lane gave a general method of studying the projective differential geometry of a configuration composed of a set of varieties generated by linear spaces with generators in correspondence, by means of the invariants and covariants of a completely integrable system of first order partial differential equations. This method was applied to quadruples of surfaces and pairs of rectilinear congruences in the doctoral dissertations of Dan Sun, 1928, and A.J. Cook, 1929, at Chicago. Other papers which employ the notion of configurations whose elements are in one-to-one correspondence are: "Ruled Surfaces with Their Generators in One-to-One Correspondence" by E.P. Lane; "Triads of Ruled Surfaces", "Complete Systems of Invariants and Covariants for Triads of Ruled Surfaces", "Tetrads of Ruled Surfaces", by A. F. Carpenter; and "Triads of Plane Curves" and "Quadruples of Space Curves" by A.R. Jerbert. The exact references to these papers will be found in the bibliography at the end of this paper.

The present paper is concerned with the geometry of a system of five three-dimensional varieties with their points in quintuple correspondence, one-to-one in every way, in a four-dimensional linear space. If we assume that the five corresponding points are linearly independent the projective theory of the configuration can be based on a system of fifteen first order partial differential equations in five dependent and three independent variables.

By suitably chosen transformations of the dependent and independent variables a canonical form of the original system of equations is established and certain invariants and covariants of this system discussed. Several associated systems are also studied.

The complications of the calculations make the study of the osculants of this configuration by the usual methods practically prohibitive. However, contact hyperquadrics are defined and conditions for their existence discussed.

Finally, each variety is considered as being referred to three one-parameter families of parametric surfaces. Since every surface in four-space sustains a conjugate net, the study of these surfaces can be based on a system of three second order partial differential equations. The varieties one family of whose parametric surfaces are the Laplace transforms of a family of surfaces on the given variety can thus be discussed.

II. DIFFERENTIAL EQUATIONS

We shall first set up a system of partial differential equations which, subject to certain integrability conditions, will represent a quintuple of three-dimensional varieties with their points in correspondence in a linear space of four dimensions S_4 , and shall indicate the transformations to be used in a study of this system.

Let us consider in a linear space S_4 five linearly independent points P_i with projective homogeneous coordinates x_k^i , ($i, k = 1, \dots, 5$). If these coordinates are analytic functions of three independent variables

$$x_k^i = x_k^i(u, v, w),$$

then, as u , v , and w range over all admissible values, the locus of each of these points is a three-dimensional variety. We have thus in the space S_4 a configuration composed of five hypersurfaces with their points in quintuple correspondence, one-to-one in every way. When we wish to indicate that the points of the quintuple are in such a relation we shall here-after denote it merely as in correspondence.

It is possible* to determine the coefficients of a system of equations of the form

*E.P.Lane, The Projective Differential Geometry of Systems of Linear Homogeneous Differential Equations of the First Order. See bibliography.

$$\begin{aligned}
 (2.1) \quad x_u^i &= a^{i\alpha} x^\alpha, \\
 x_v &= b^{i\alpha} x^\alpha, \\
 x_w &= c^{i\alpha} x^\alpha, \quad (1, \alpha = 1, \dots, 5),
 \end{aligned}$$

where the summation convention of tensor analysis has been introduced*, and where the subscripts denote partial differentiation, so that the functions x^i will be five sets of solutions. That is, we may substitute each of these sets in turn in each of the equations of (2.1) and solve the resulting systems of equations for the coefficients. Unless otherwise stated the superscripts and subscripts $i, j, k, \alpha, \beta, \gamma$ will be understood to range over the values $1, \dots, 5$ throughout this paper.

The system (2.1) will be completely integrable in case the coefficients satisfy the following integrability conditions, obtained by demanding that the second derivatives of the x^i be unique,

$$\begin{aligned}
 (2.2) \quad a_v^{ij} + a^{i\alpha} b^{j\alpha} &= b_u^{ij} + b^{i\alpha} a^{j\alpha}, \\
 a_w^{ij} + a^{i\alpha} c^{j\alpha} &= c_u^{ij} + c^{i\alpha} a^{j\alpha}, \\
 b_w^{ij} + b^{i\alpha} c^{j\alpha} &= c_v^{ij} + c^{i\alpha} b^{j\alpha}.
 \end{aligned}$$

One finds consequently

$$(2.3) \quad \sum_i a_v^{ii} = \sum_i b_u^{ii}, \quad \sum_i a_w^{ii} = \sum_i c_u^{ii}, \quad \sum_i b_w^{ii} = \sum_i c_v^{ii}.$$

From this last set of equations we can see that there exists a function $g(u, v, w)$ such that

*A.J.Cook, Pairs of Rectilinear Congruences. See bibliography

$$\sum_i a^{ii} = g_u, \quad \sum_i b^{ii} = g_v, \quad \sum_i c^{ii} = g_w,$$

and hence

$$\Delta \equiv (x^1, x^2, x^3, x^4, x^5) \quad ce^g,$$

where $c = \text{const.} \neq 0$. If $\Delta \equiv 0$, the corresponding points P_i of the hypersurfaces lie in the same hyperplane, and in this case we shall say that the quintuple is degenerate.

The most general set of solutions of (2.1) is a linear combination with constant coefficients of any particular set. That is, the most general quintuple of integral varieties V_3 determined by the equations (2.1) and (2.2) is a projective transform of any particular one. Hence, only the projective properties of this system can be studied by this method.

The transformation of the dependent variables

$$(2.4) \quad x^i = \lambda_i \bar{x}^i,$$

where the λ_i are functions of the three independent variables, and the transformation of the independent variables

$$(2.5) \quad u = \phi(u, v, w), \quad v = \psi(u, v, w), \quad w = \theta(u, v, w).$$

$$J(\phi, \psi, \theta) \equiv \frac{\partial(\phi, \psi, \theta)}{\partial(u, v, w)} \neq 0$$

leave each of the varieties V_3 invariant and are the most general transformations that leave the form of the differential equations (2.1) unaltered. Therefore, the projective differential geometry of a quintuple of hypersurfaces in the space S_4 , with their points in correspondence, can be studied

by means of the invariants and covariants of the system
(2.1) under the total transformation (2.4), (2.5). The
details of carrying out these transformations will be given
in the following section.

III. A CANONICAL FORM AND INVARIANTS

By applying the transformations noted at the end of the preceding section to the general system of equations and placing certain conditions on the new coefficients we arrive in this section at a canonical form of our system of equations, the use of which will simplify the calculations later in the paper.

Upon applying the transformation

$$(2.4) \quad x^i = \lambda_i \bar{x}^i$$

to the general system of equations (2.1) we arrive at another system of the same type

$$(3.1) \quad \begin{aligned} \bar{x}_u^i &= a^{i\alpha} \bar{x}^\alpha, \\ \bar{x}_v^i &= b^{i\alpha} \bar{x}^\alpha, \\ \bar{x}_w^i &= c^{i\alpha} \bar{x}^\alpha \end{aligned} \quad (i, \alpha = 1, \dots, 5),$$

where

$$(3.2) \quad \bar{a}^{ij} = \frac{\lambda_j}{\lambda_i} a^{ij}, \quad \bar{b}^{ij} = \frac{\lambda_j}{\lambda_i} b^{ij}, \quad \bar{c}^{ij} = \frac{\lambda_j}{\lambda_i} c^{ij}, \quad (i \neq j),$$

and

$$(3.3) \quad \bar{a}^{ii} = a^{ii} - \frac{\partial}{\partial u} \log \lambda_i, \quad \bar{b}^{ii} = b^{ii} - \frac{\partial}{\partial v} \log \lambda_i, \quad \bar{c}^{ii} = c^{ii} - \frac{\partial}{\partial w} \log \lambda_i.$$

We shall now show that it is possible to obtain a canonical form of these equations characterized by the conditions

$$(3.4) \quad \bar{c}^{ii} - \bar{b}^{ii} = \bar{b}^{ii} - \bar{a}^{ii} = \sum_i \bar{a}^{ii} = 0.$$

The first two of these equations impose on the functions λ_i the conditions

$$(3.5) \quad \begin{aligned} \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v} \right) \log \lambda_i &= c^{ii} - b^{ii}, \\ \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u} \right) \log \lambda_i &= b^{ii} - a^{ii}. \end{aligned}$$

A system of differential equations of the form (2.1) can always be transformed into another system of the same type characterized by the first two relations of (3.4). We shall say that as a result of this transformation the system (2.1) has been reduced to its intermediate form

$$(3.7) \quad \begin{aligned} x_u^i &= a^{i\alpha} x^\alpha, \\ x_v^i &= b^{i\alpha} x^\alpha, \\ x_w^i &= c^{i\alpha} x^\alpha, \\ c^{ii} - b^{ii} &= b^{ii} - a^{ii} = 0 \quad (i, \alpha = 1, \dots, 5). \end{aligned}$$

The most general transformation which leaves the intermediate form invariant may be found by applying the transformation (2.4) to the intermediate form. The result will be in the same form in case

$$(3.8) \quad \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v} \right) \log \lambda_i = \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u} \right) \log \lambda_i = 0.$$

This is a Jacobian* system whose solutions are

$$\lambda_i = f_i(u+v+w),$$

where the f_i are arbitrary functions of $u+v+w$. Hence, the most general transformation of the dependent variable preserving the intermediate form is

$$(3.9) \quad x^i = e^{f_i(u+v+w)} \bar{x}^i.$$

*Goursat-Hedrick, Mathematical Analysis, II, 2, p. 267.

From this form we may obtain the canonical form for which not only $a^{ii} - b^{ii} = b^{ii} - a^{ii} = 0$, but also $\sum_i a^{ii} = 0$.

Applying the transformation (3.9) to the intermediate form and denoting the new coefficients by $\bar{a}^{ij}$, $\bar{b}^{ij}$, and $\bar{c}^{ij}$ we find

$$\begin{aligned}\bar{a}^{ii} &= \bar{b}^{ii} = \bar{c}^{ii} = a^{ii} - f_i', \\ \bar{a}^{ij} &= e^{f_i - f_j} a^{ij}, \quad \bar{b}^{ij} = e^{f_j - f_i} b^{ij}, \quad \bar{c}^{ij} = e^{f_j - f_i} c^{ij},\end{aligned}$$

where the accent denotes differentiation with respect to $u+v+w$. From (2.3) we have

$$\sum_i a^{ii} = g_u, \quad \sum_i b^{ii} = g_v, \quad \sum_i c^{ii} = g_w,$$

and hence

$$\sum_i \bar{a}^{ii} = \sum_i \bar{b}^{ii} = \sum_i \bar{c}^{ii} = g'(u+v+w) = 0.$$

Then from the third condition of (3.4)

$$\sum_i \bar{a}^{ii} = \sum_i \bar{b}^{ii} = \sum_i \bar{c}^{ii} = g'(u+v+w) + \sum_i f_i'(u+v+w) = 0.$$

Thus we may select f_i so that

$$\sum_i f_i (u+v+w) = g (u+v+w) + \text{const.}$$

Therefore any system in the intermediate form (3.7) can be transformed into another in the canonical form

$$\begin{aligned}(3.10) \quad x_u^i &= a^{i\alpha} x^\alpha, \\ x_v^i &= b^{i\alpha} x^\alpha, \\ x_w^i &= c^{i\alpha} x^\alpha, \\ a^{ii} - b^{ii} &= b^{ii} - a^{ii} = \sum_i a^{ii} = 0 \quad (i, \alpha = 1, \dots, 5).\end{aligned}$$

The transformation employed in this reduction is

$$\begin{aligned}(3.11) \quad x^i &= e^{f_i(u+v+w)} \bar{x}^i, \\ \sum_i f_i (u+v+w) &= g (u+v+w) + \text{const.}\end{aligned}$$

Upon applying the transformation (3.11) to the canonical system (3.10) we find that the most general transformation of the dependent variable which leaves the canonical form invariant is

$$x^i = e^{f_i(u+v+w)} \bar{x}^i, \\ \sum_i f_i(u+v+w) = \text{const.}$$

It is evident that the a^{ij} , b^{ij} , c^{ij} , ($i, j = 1, \dots, 5$), are seventy-five relative seminvariants of the system (3.10) under the transformation (3.12) and that

$$\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v}\right) \bar{a}^{ii} = \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v}\right) a^{ii}, \\ \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right) \bar{a}^{ii} = \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right) a^{ii}.$$

Hence $\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial v}\right) a^{ii}$ and $\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right) a^{ii}$ are absolute seminvariants. However, the first two conditions of the canonical form together with the integrability conditions (2.2) show that these seminvariants are dependent upon the relative seminvariants. We may also note that we can solve the integrability conditions (2.2) for any thirty of these seminvariants in terms of the forty-five others, say the a^{ii} , b^{ii} , c^{ii} , and a^{i+i} , b^{i+i} , c^{i+i} . Then we will have as a fundamental set of seminvariants

a^{ij} , b^{ij} , c^{ij} , ($i, j = 1, \dots, 5; i+1 = 2, 3, 4, 5, 1; i \neq j; i+1 \neq j$). The fundamental semicovariants are the x^i .

Let us next consider the effect of the change of the independent variable on the canonical system (3.10). Applying the transformation

(2.5) $u = \phi(u, v, w)$, $v = \psi(u, v, w)$, $w = \theta(u, v, w)$, $J(\phi, \psi, \theta) \neq 0$
to the system (3.10), we again arrive at a system of the same
form where the new coefficients $\bar{a}^{ij}$, $\bar{b}^{ij}$, $\bar{c}^{ij}$ are given by

$$J \cdot \bar{a}^{ij} = \frac{\partial(\psi, \theta)}{\partial(v, w)} a^{ij} - \frac{\partial(\psi, \theta)}{\partial(u, w)} b^{ij} + \frac{\partial(\psi, \theta)}{\partial(u, v)} c^{ij},$$

$$J \cdot \bar{b}^{ij} = -\frac{\partial(\phi, \theta)}{\partial(v, w)} a^{ij} + \frac{\partial(\phi, \theta)}{\partial(u, w)} b^{ij} - \frac{\partial(\phi, \theta)}{\partial(u, v)} c^{ij},$$

$$J \cdot \bar{c}^{ij} = \frac{\partial(\phi, \psi)}{\partial(v, w)} a^{ij} - \frac{\partial(\phi, \psi)}{\partial(u, w)} b^{ij} + \frac{\partial(\phi, \psi)}{\partial(u, v)} c^{ij} \quad (i \neq j),$$

and

$$J \cdot \bar{a}^{ii} = \left(\frac{\partial(\psi, \theta)}{\partial(v, w)} - \frac{\partial(\psi, \theta)}{\partial(u, w)} + \frac{\partial(\psi, \theta)}{\partial(u, v)} \right) a^{ii},$$

$$J \cdot \bar{b}^{ii} = -\left(\frac{\partial(\phi, \theta)}{\partial(v, w)} + \frac{\partial(\phi, \theta)}{\partial(u, w)} - \frac{\partial(\phi, \theta)}{\partial(u, v)} \right) a^{ii},$$

$$J \cdot \bar{c}^{ii} = \left(\frac{\partial(\phi, \psi)}{\partial(v, w)} - \frac{\partial(\phi, \psi)}{\partial(u, w)} + \frac{\partial(\phi, \psi)}{\partial(u, v)} \right) a^{ii} \quad (i, j = 1, \dots, 5).$$

We have also the canonical conditions

$$\bar{c}^{ii} - \bar{b}^{ii} = \frac{\bar{a}^{ii}}{J} \begin{vmatrix} 1 & \phi_u & \psi_u & + & \theta_u \\ 1 & \phi_v & \psi_v & + & \theta_v \\ 1 & \phi_w & \psi_w & + & \theta_w \end{vmatrix} = 0,$$

$$\bar{b}^{ii} - \bar{a}^{ii} = \frac{\bar{a}^{ii}}{J} \begin{vmatrix} 1 & \phi_u & + & \psi_u & & \theta_u \\ 1 & \phi_v & + & \psi_v & & \theta_v \\ 1 & \phi_w & + & \psi_w & & \theta_w \end{vmatrix} = 0,$$

and

$$\sum_i \bar{a}^{ii} = \frac{1}{J} \begin{vmatrix} 1 & \psi_u & \theta_u \\ 1 & \psi_v & \theta_v \\ 1 & \psi_w & \theta_w \end{vmatrix} \cdot \sum_i a^{ii} = 0,$$

$$\sum_i \bar{b}^{ii} = \frac{1}{J} \begin{vmatrix} 1 & \phi_u & \theta_u \\ 1 & \phi_v & \theta_v \\ 1 & \phi_w & \theta_w \end{vmatrix} \cdot \sum_i a^{ii} = 0,$$

$$\sum_i \bar{c}^{ii} = \frac{1}{J} \begin{vmatrix} 1 & \phi_u & \psi_u \\ 1 & \phi_v & \psi_v \\ 1 & \phi_w & \psi_w \end{vmatrix} \cdot \sum_i a^{ii} = 0,$$

It can be verified that the most general transformation of the independent variable which leaves the canonical system in the same form is characterized by the condition

$$\phi + \psi + \theta = \alpha(u+v+w),$$

where α is an arbitrary function of $u+v+w$. The canonical transformation is then

$$(3.13) \quad \bar{u} = \phi(u, v, w), \quad \bar{v} = \psi(u, v, w), \quad \bar{w} = \alpha(u+v+w) - \phi - \psi,$$

$$J(, , -) = \alpha' \begin{vmatrix} \phi_u & \psi_u & 1 \\ \phi_v & \psi_v & 1 \\ \phi_w & \psi_w & 1 \end{vmatrix} = \alpha' \Delta \neq 0,$$

where

$$\alpha' = \frac{d\alpha}{d(u+v+w)},$$

and hence

$$\bar{a}^{ii} = \bar{b}^{ii} = \bar{c}^{ii} = \frac{1}{\alpha'} a^{ii},$$

$$\bar{a}^{ij} = \frac{1}{J} \left[\begin{vmatrix} \psi_v & \alpha' - \phi_v \\ \psi_w & \alpha' - \phi_w \end{vmatrix} a^{ij} - \begin{vmatrix} \psi_u & \alpha' - \phi_u \\ \psi_w & \alpha' - \phi_w \end{vmatrix} b^{ij} + \begin{vmatrix} \psi_u & \alpha' - \phi_u \\ \psi_v & \alpha' - \phi_v \end{vmatrix} c^{ij} \right]$$

$$\bar{b}^{ij} = \frac{1}{J} \left[\begin{vmatrix} \phi_v & \alpha' - \psi_v \\ \phi_w & \alpha' - \psi_w \end{vmatrix} a^{ij} - \begin{vmatrix} \phi_u & \alpha' - \psi_u \\ \phi_w & \alpha' - \psi_w \end{vmatrix} b^{ij} + \begin{vmatrix} \phi_u & \alpha' - \psi_u \\ \phi_v & \alpha' - \psi_v \end{vmatrix} c^{ij} \right]$$

$$\bar{c}^{ij} = \frac{1}{J} \left[\begin{vmatrix} \phi_v & \psi_v \\ \phi_w & \psi_w \end{vmatrix} a^{ij} - \begin{vmatrix} \phi_u & \psi_u \\ \phi_w & \psi_w \end{vmatrix} b^{ij} + \begin{vmatrix} \phi_u & \psi_u \\ \phi_v & \psi_v \end{vmatrix} c^{ij} \right]$$

It is easy to verify that

$$(3.14) \quad d^{ij} = \sum_{\kappa} (-1)^{i+\kappa} a^{i\kappa} (b^{il} c^{jh} - b^{ih} c^{jl})$$

(i, j, k, l, h = 1, ..., 5; k ≠ i, j; l, h ≠ i, j, k; l > h)

are twenty relative invariants under this transformation.

In fact

$$\bar{d}^{ij} = \frac{1}{\alpha' \Delta} d^{ij} = \frac{1}{J} d^{ij}.$$

Also, it is obvious that the five covariants x^i are a
fundamental set of covariants.

IV. POWER SERIES EXPANSIONS

We shall now set up the power series expansions for a point $P_{\bar{x}}$ near P_i on any one of the original hypersurfaces. These expansions will be found useful in determining the contact hyperquadrics of Section VIII.

Take any five points $P_1, \dots, P_5$, which according to the hypothesis of Section I do not lie in the same space S_3 , as vertices of a local five-point of reference. Consider a point $P_{\bar{x}}$ whose coordinates referred to this local five-point are given by $y_i = \lambda_i$. There will be five values of z^i corresponding to the five sets of y_k^i . These are given by

$$z = \lambda^\alpha y^\alpha,$$

where the superscripts and subscripts have been dropped.

Any point $P_{\bar{x}}$ near P_i on the variety determined by x^i will be given by

$$\begin{aligned} X = x^i &+ (x_u^i \Delta u + x_v^i \Delta v + x_w^i \Delta w) \\ (4.1) \quad &+ \frac{1}{2}(x_{uu}^i \Delta u^2 + x_{vv}^i \Delta v^2 + x_{ww}^i \Delta w^2 \\ &+ 2x_{uv}^i \Delta u \Delta v + 2x_{uw}^i \Delta u \Delta w + 2x_{vw}^i \Delta v \Delta w) \\ &+ \dots, \end{aligned}$$

where the second order derivatives are given by

$$\begin{aligned} x_{uu}^i &= (a^{i\beta} + a^{i\alpha} a^{\alpha\beta}) x^\beta, \\ x_{vv}^i &= (b^{i\beta} + b^{i\alpha} b^{\alpha\beta}) x^\beta, \\ x_{ww}^i &= (c^{i\beta} + c^{i\alpha} c^{\alpha\beta}) x^\beta, \\ (4.2) \quad x_{uv}^i &= (a^{i\beta} + a^{i\alpha} b^{\alpha\beta}) x^\beta, \\ x_{uw}^i &= (a^{i\beta} + a^{i\alpha} c^{\alpha\beta}) x^\beta, \\ x_{vw}^i &= (b^{i\beta} + b^{i\alpha} c^{\alpha\beta}) x^\beta \quad (1, \alpha, \beta = 1, \dots, 5). \end{aligned}$$

Substituting these values for the second derivatives and those for the first derivatives from the original system in the equation (4.1) and collecting the coefficients of the x^i , we have

$$x^i = y_{\alpha}^i x^{\alpha} \quad (i, \alpha = 1, \dots, 5).$$

where the y_j^i are the coordinates of the point $P_{\underline{x}}$ referred to the local five-point of reference $P_1, \dots, P_5$. The local coordinates of a point $P_{\underline{x}}$ near P_i are, therefore, given by the power series

$$\begin{aligned} y_j^i = & \delta_j^i + a^{ij} \Delta u + b^{ij} \Delta v + c^{ij} \Delta w \\ & + \frac{1}{2} [(a_{\underline{u}}^{ij} + a^{i\alpha} a^{\alpha j}) \Delta u^2 + (b_{\underline{v}}^{ij} + b^{i\alpha} b^{\alpha j}) \Delta v^2 \\ & + (c_{\underline{w}}^{ij} + c^{i\alpha} c^{\alpha j}) \Delta w^2 + 2(a_{\underline{v}}^{ij} + a^{i\alpha} b^{\alpha j}) \Delta u \Delta v \\ & + 2(a_{\underline{w}}^{ij} + a^{i\alpha} c^{\alpha j}) \Delta u \Delta w + 2(b_{\underline{w}}^{ij} + b^{i\alpha} c^{\alpha j}) \Delta v \Delta w \\ & + \dots, \end{aligned}$$

where δ_j^i is the Kronecker delta. These are the series which we started out to obtain.

V. AN ADJOINT SYSTEM

If we consider each hypersurface as the envelope of a three-parameter family of tangent hyperplanes instead of the locus of ∞^3 points, we will arrive at a dual system of equations. That is, we will have a new system of equations using hyperplanes instead of point coordinates. The quintuple of coordinates determining respectively the tangent hyperplanes at the points P_i will form a fundamental system of solutions of this new system of equations which we shall call the adjoint of the canonical system.

The equation of the tangent hyperplane of the variety V_3 determined by x^i at the point P_i is

$$(y^i, x^i, x_{\alpha}^i, x_{\beta}^i, x_{\omega}^i) = 0,$$

and the coordinates of this hyperplane will be proportional to the cofactors of the y^i in the above determinant. If we denote these cofactors by λ^i we have

$$\rho \cdot \lambda^i = (x^i, x_{\alpha}^i, x_{\beta}^i, x_{\omega}^i).$$

In the following pages the proportionality factor will be omitted. Substituting for the partial derivatives from our original system we have, except for a proportionality factor,

$$(5.1) \quad \lambda^i = d^i \eta^{\alpha} \quad (\alpha, i = 1, \dots, 5; \alpha \neq i),$$

where

$$(5.2) \quad \begin{aligned} \eta^1 &= (x^1, x^3, x^4, x^5), & \eta^2 &= (x^1, x^4, x^5, x^1), \\ \eta^3 &= (x^4, x^5, x^1, x^2), & \eta^4 &= (x^5, x^1, x^2, x^3), \\ \eta^5 &= (x^1, x^2, x^3, x^4), \end{aligned}$$

and the d^{ij} are given by (3.14).

Solving the equations (5.1) for η^i we obtain

$$(5.3) \quad d \cdot \eta^i = D^{i\alpha} \lambda^\alpha,$$

where d is the determinant of the d^{ij} and the D^{ji} are the cofactors of d^{ij} in the determinant d , which does not vanish identically in u , v , and w for arbitrary coefficients of the canonical system (3.10)

Differentiating (5.1) and making use of (3.10) we find

$$(5.4) \quad \lambda_u^i = e^{i\alpha} \eta^\alpha, \quad \lambda_v^i = f^{i\alpha} \eta^\alpha, \quad \lambda_w^i = g^{i\alpha} \eta^\alpha,$$

where

$$\begin{aligned} e^{ij} &= (-1)^{i+j+1} (d_{ij}^{ij} - d^{i\alpha} a^{j\alpha}), \\ f^{ij} &= (-1)^{i+j+1} (d_{ij}^{ij} - d^{i\alpha} b^{j\alpha}), \\ g^{ij} &= (-1)^{i+j+1} (d_{ij}^{ij} - d^{i\alpha} c^{j\alpha}) \quad (i \neq \alpha, j). \end{aligned}$$

When we eliminate the η^i between (5.3) and (5.4) we obtain the sytem of differential equations of the quintuple of hypersurfaces in hyperplane coordinates.

$$(5.6) \quad \lambda_u^i = A^{i\alpha} \lambda^\alpha, \quad \lambda_v^i = B^{i\alpha} \lambda^\alpha, \quad \lambda_w^i = C^{i\alpha} \lambda^\alpha \quad (i, \alpha = 1, \dots, 5),$$

where

$$(5.7) \quad d \cdot A^{ij} = e^{i\alpha} B^{j\alpha}, \quad d \cdot B^{ij} = f^{i\alpha} D^{j\alpha}, \quad d \cdot C^{ij} = g^{i\alpha} D^{j\alpha} \quad (i \neq \alpha).$$

For this system the canonical conditions

$$C^{ii} - B^{ii} = B^{ii} - A^{ii} = \sum_i A^{ii} = 0$$

are not identically satisfied. The analytic conditions that they shall be satisfied are

$$\begin{aligned} d \cdot (C^{ii} - B^{ii}) &= (-1)^{i+\alpha+i'} D^{i\alpha} (d_{\omega}^{i\alpha} - d_{\nu}^{i\alpha} - d^{i\beta} c^{\alpha\beta} + d^{i\beta} b^{\alpha\beta}) = 0, \\ d \cdot (B^{ii} - A^{ii}) &= (-1)^{i+\alpha+i'} D^{i\alpha} (d_{\nu}^{i\alpha} - d_{\omega}^{i\alpha} - d^{i\beta} b^{\alpha\beta} + d^{i\beta} a^{\alpha\beta}) = 0, \\ d \cdot \sum_i A^{ii} &= \sum_i (-1)^{i+\alpha+i'} D^{i\alpha} (d_{\omega}^{i\alpha} - d^{i\beta} a^{\alpha\beta}) = 0. \end{aligned}$$

The geometrical restriction placed upon the fundamental system of hypersurfaces by these conditions and the consideration of the self-adjoint problem will not be taken up in this paper.

VI. A RECIPROCAL SYSTEM

When we consider in this section the quintuple of varieties enveloped by the hyperplanes of the local five-point of reference we find that the resulting system is reciprocal to the original system. That is, the canonical system of equations can be formed from the new system in the same manner that the new system is formed from the canonical system.

The equation of the hyperplane opposite the vertex P_1 in the five-point of reference, i.e. the hyperplane determined by the points $P_2, \dots, P_5$, is

$$(y, x^1, x^2, x^4, x^5) = 0.$$

The coordinates of this hyperplane will be proportional to the cofactors of the y_i of this determinant. Similarly, we find the coordinates of the remaining four hyperplanes. They are precisely the η^i of Section V which are given by (5.2).

Differentiating these determinants and making use of (3.10), we find the differential equations for this quintuple of varieties to be

$$\begin{aligned} \eta_u^i &= (-1)^{i+\alpha+1} a^{\alpha i} \eta^\alpha, \\ \eta_v^i &= (-1)^{i+\alpha+1} b^{\alpha i} \eta^\alpha, \\ \eta_w^i &= (-1)^{i+\alpha+1} c^{\alpha i} \eta^\alpha, \end{aligned} \quad (6.1)$$

$$c^{ii} - b^{ii} = b^{ii} - a^{ii} = \sum_i a^{ii} = 0 \quad (1, \alpha = 1, \dots, 5).$$

which is in its canonical form.

When we compare this system of equations with the canonical system (3.10) we see that the system (3.10) can be formed from (6.1) in the same way that (6.1) was formed from (3.10). That is, the relation between the canonical system (3.10) and the associated system (6.1) is a reciprocal one.

However, it is not necessary for the original system (2.1) to be in the canonical form for the two systems to be reciprocal. If we repeat the process starting with (2.1) we find that a necessary and sufficient condition for the system of equations representing the quintuple of varieties enveloped by the hyperplanes of the five-point of reference to be reciprocal to the original system (2.1) is a form such that

$$\sum_i a^{ii} = \sum_i b^{ii} = \sum_i c^{ii} = 0.$$

The reciprocal system coincides with the original system in case

$$\begin{aligned} a^{ii} &= b^{ii} = c^{ii} = 0, \\ a^{ij} &= (-1)^{i+j+1} a^{ji}, \\ b^{ij} &= (-1)^{i+j+1} b^{ji}, \\ c^{ij} &= (-1)^{i+j+1} c^{ji} \quad (i, j = 1, \dots, 5; j \neq i). \end{aligned}$$

VII. ASSOCIATE QUINTUPLES

In this section we shall determine the point transformations of two quintuples of hypersurfaces associated with the original quintuple. The one, which we shall call the minus associate transform,* was defined in Section VI as the envelope of the hyperplanes of the local five-point of reference. The other, the first associate transform,* is the loci of the five points of intersection of the tangent hyperplanes of the original five varieties at corresponding points P_i .

In section VI we defined the minus associate transform as the envelope of the hyperplanes of the local five-point of reference and found its system of equations in hyperplane coordinates. We shall now determine the five points of contact of these hyperplanes with the corresponding varieties.

P_j , for example, is the intersection of the four hyperplanes determined by the points $x^j(u, v, w)$, $x^j(u + \Delta u, v, w)$, $x^j(u, v + \Delta v, w)$, and $x^j(u, v, w + \Delta w)$, ($j = 2, \dots, 5$), respectively, as Δu , Δv , and Δw simultaneously approach zero. We have by Taylor's theorem

*D. Sun, Quadruples of Surfaces. See bibliography.

$$\begin{aligned}
 x(u+\Delta u, v, w) &= x^i + x_a^i \Delta u + \dots, \\
 &= x^i + a^{i\alpha} x^\alpha \Delta u + \dots, \\
 x(u, v+\Delta v, w) &= x^i + b^{i\alpha} x^\alpha \Delta v + \dots, \\
 x(u, v, w+\Delta w) &= x^i + c^{i\alpha} x^\alpha \Delta w + \dots \quad (i, \alpha = 1, \dots, 5).
 \end{aligned}$$

Any point on the hyperplane determined by $x(u+\Delta u, v, w)$, $(i = 2, \dots, 5)$, is given by

$$\begin{aligned}
 &\lambda x^2(u+\Delta u, v, w) + \mu x^3(u+\Delta u, v, w) + \nu x^4(u+\Delta u, v, w) + \rho x^5(u+\Delta u, v, w) \\
 &= x^i \left(\lambda a^{i1} + \mu a^{i2} + \nu a^{i3} + \rho a^{i4} \right) \Delta u \\
 &+ x^i \left[\lambda + \left(\lambda a^{i2} + \mu a^{i3} + \nu a^{i4} + \rho a^{i5} \right) \Delta u \right] \\
 &+ x^i \left[\mu + \left(\lambda a^{i3} + \mu a^{i4} + \nu a^{i5} + \rho a^{i6} \right) \Delta u \right] \\
 &+ x^i \left[\nu + \left(\lambda a^{i4} + \mu a^{i5} + \nu a^{i6} + \rho a^{i7} \right) \Delta u \right] \\
 &+ x^i \left[\rho + \left(\lambda a^{i5} + \mu a^{i6} + \nu a^{i7} + \rho a^{i8} \right) \Delta u \right]
 \end{aligned}$$

Similar equations may be obtained for the points on the hyperplanes determined by $x(u, v+\Delta v, w)$ and $x(u, v, w+\Delta w)$ respectively by the substitution

$$\begin{pmatrix} u & a & \lambda & \mu & \nu & \rho \\ v & b & \lambda' & \mu' & \nu' & \rho' \\ w & c & \lambda'' & \mu'' & \nu'' & \rho'' \end{pmatrix}$$

Any point on the hyperplane determined by the points $P_2, \dots, P_5$ is given by

$$\lambda'' x^2 + \mu'' x^3 + \nu'' x^4 + \rho'' x^5.$$

For the point $\bar{P}_i$ of intersection of these four hyperplanes as Δu , Δv , and Δw simultaneously approach zero we have

$$\lambda a^{2'} + \mu a^{3'} + \nu a^{4'} + \rho a^{5'} = 0,$$

$$\lambda' b^{2'} + \mu' b^{3'} + \nu' b^{4'} + \rho' b^{5'} = 0,$$

$$\lambda'' c^{2'} + \mu'' c^{3'} + \nu'' c^{4'} + \rho'' c^{5'} = 0,$$

$$\lambda = k^1 \lambda' = k^{1''} \lambda'' = k^{1'''} \lambda''',$$

$$\mu = k^1 \mu' = k^{1''} \mu'' = k^{1'''} \mu''',$$

$$\nu = k^1 \nu' = k^{1''} \nu'' = k^{1'''} \nu''',$$

$$\rho = k^1 \rho' = k^{1''} \rho'' = k^{1'''} \rho''',$$

where the $k^1, k^{1''}, k^{1'''}$ are factors of proportionality. Solving these equations for λ, μ, ν , and ρ we have, except for a proportionality factor

$$\lambda = \mathcal{J}^{12}, \mu = \mathcal{J}^{13}, \nu = \mathcal{J}^{14}, \rho = \mathcal{J}^{15},$$

where

$$(7.1) \quad \mathcal{J}^{ij} = \sum_k (-1)^{i+k} a^{ki} (b^{hi} c^{li} - b^{li} c^{hi})$$

($i, j, k, h, l = 1, \dots, 5; i \neq j; k \neq i, j; l, h \neq i, j, k; l > h$).

hence the point $\bar{P}_i$ is given by

$$\rho \cdot \bar{x}^i = \mathcal{J}^{i\alpha} x^\alpha, \quad (\alpha = 2, \dots, 5).$$

Similarly we may find the points $P, \dots, P$.

Hence, the points of contact of the hyperplanes of the five-point of reference with the varieties of the minus associate quintuple are

$$(7.2) \quad \bar{x}^i = \mathcal{J}^{i\alpha} x^\alpha, \quad (i, \alpha = 1, \dots, 5; i \neq \alpha).$$

The $\mathcal{J}^{ij}$ are obviously relative invariants of the canonical system under the transformation (3.13). In fact

$$\bar{\mathcal{J}}^{ij} = \frac{1}{\alpha' \Delta} \mathcal{J}^{ij},$$

Hence the $\bar{x}^i$ are five relative covariants of the canonical system.

It may be noted that if the determinant J of the J^{ii} vanishes identically the five points $\bar{x}^i$ lie in the same hyperplane and the system is degenerate.

We again remark that the minus associate transform of the minus associate transform of the original quintuple is again the original system. That is, the relation is reciprocal.

It is readily seen that the five corresponding tangent hyperplanes at the points P_i intersect in groups of four in five points, in general linearly independent, which form a complete space five-point in S_4 . We shall denote these points by P_{ji} , where each P_{ji} is the intersection of the four tangent hyperplanes at P_j , ($i \neq j$). The coordinates of P_{ji} referred to the local five-point of reference $P_1, \dots, P_5$ are given by D^{ij} , where the D^{ij} are those defined in Section V. Hence we have a first associate quintuple in this way related to the original system.

The corresponding points P_{ji} of the first associate quintuple are given by the equations

$$(7.3) \quad c \cdot y^i = D^{i\alpha} x^\alpha, \quad (i, \alpha = 1, \dots, 5).$$

Differentiating these equations and eliminating the x^i we arrive at the system of differential equations for the first associate quintuple

$$(7.4) \quad \begin{aligned} y_{\alpha}^i &= A^{i\alpha} y^{\alpha}, \\ y_{\beta}^i &= B^{i\alpha} y^{\alpha}, \\ y_{\gamma}^i &= C^{i\alpha} y^{\alpha}, \end{aligned}$$

subject to integrability conditions which are the same as those for the original system with the substitution

$$\begin{pmatrix} a & b & c \\ A & B & C \end{pmatrix},$$

where the A^{ij} , B^{ij} , and C^{ij} are the coefficients obtained by solving the equation

$$\begin{vmatrix} y_{\alpha}^i & D_{\alpha}^{i1} + a^{1\alpha} D^{i\alpha} & D_{\alpha}^{i2} & a^{2\alpha} D^{i\alpha} & D_{\alpha}^{i3} + a^{3\alpha} D^{i\alpha} & D_{\alpha}^{i4} + a^{4\alpha} D^{i\alpha} & D_{\alpha}^{i5} + a^{5\alpha} D^{i\alpha} \\ y^1 & D^{11} & D^{12} & D^{13} & D^{14} & D^{15} \\ y^2 & D^{21} & D^{22} & D^{23} & D^{24} & D^{25} \\ y^3 & D^{31} & D^{32} & D^{33} & D^{34} & D^{35} \\ y^4 & D^{41} & D^{42} & D^{43} & D^{44} & D^{45} \\ y^5 & D^{51} & D^{52} & D^{53} & D^{54} & D^{55} \end{vmatrix} = 0$$

for y_{α}^i , and two similar equations for y_{β}^i and y_{γ}^i , respectively, in terms of $y^1, \dots, y^5$.

If the $D^{ij} = 0$, ($i \neq j$), that is, if the first associate system coincides with the original system, the five-points $P_1, \dots, P_5$ and $P_{y_1}, \dots, P_{y_5}$ coincide and P_i lies in the hyperplane determined by P_j, P_k, P_l, P_h , ($i \neq j \neq k \neq l \neq h$), and the original system is degenerate.

Therefore, if we continue taking the first associate transforms of the first associate transforms and in this way obtain an associate sequence of transforms, we may state that

the period of the first associate sequence is always ≥ 1 if the original system is non-degenerate.

If the minus associate transform and the first associate transform coincide the coefficients of (7.2) must be proportional to those of (7.3) and hence the minus associate transform coincides with the first associate transform, that is, the five point $\bar{P}_i$ coincides with the five point P_{y_i} , in case

$$D^{ii} = 0,$$

$$D^{il} : D^{ik} : D^{ih} = f^{il} : f^{ik} ; f^{ih}$$

$$(i, l, k, h = 1, \dots, 5; l \neq k \neq h \neq i).$$

VIII. CONTACT QUADRIC HYPERSURFACES

In this section we shall consider various quadric varieties V_3^i having contact of first, or higher, order with the fundamental system of varieties V_3 at the corresponding points P_i . A hyperquadric is said to have contact of order n with a variety V_3 at a point P in case every curve on this variety through the point P has $n+1$ coincident points in common with the hyperquadric at the point P .

The most general quadric hypersurface in the space S_4 has the equation

$$(8.1) \quad k^{\alpha\beta} y^\alpha y^\beta = 0 \quad (\alpha, \beta = 1, \dots, 5; \alpha < \beta).$$

There exist ∞^4 of these hyperquadrics, of which ∞^1 pass through the five points P_i . The most general hyperquadric passing through the five points P_i has the equation

$$(8.2) \quad k^{\alpha\beta} y^\alpha y^\beta = 0 \quad (\alpha \neq \beta; \alpha < \beta).$$

If we substitute the power series expansions for y^i from (4.2) in this last equation and demand that the hyperquadric have first order contact at each of the points P_i , we arrive at the fifteen conditions

$$\begin{aligned}
 & a'^2 k'^2 + a'^3 k'^3 + a'^4 k'^4 + a'^5 k'^5 = 0, \\
 & b'^2 k'^2 + b'^3 k'^3 + b'^4 k'^4 + b'^5 k'^5 = 0, \\
 & c'^2 k'^2 + c'^3 k'^3 + c'^4 k'^4 + c'^5 k'^5 = 0, \\
 & a'^2 k'^2 + a'^1 k'^1 + a'^4 k'^4 + a'^5 k'^5 = 0, \\
 & b'^2 k'^2 + b'^3 k'^3 + b'^4 k'^4 + b'^5 k'^5 = 0, \\
 & c'^2 k'^2 + c'^3 k'^3 + c'^4 k'^4 + c'^5 k'^5 = 0, \\
 (8.3) \quad & a'^1 k'^3 + a'^2 k'^2 + a'^4 k'^4 + a'^5 k'^5 = 0, \\
 & b'^1 k'^3 + b'^2 k'^2 + b'^4 k'^4 + b'^5 k'^5 = 0, \\
 & c'^1 k'^3 + c'^2 k'^2 + c'^4 k'^4 + c'^5 k'^5 = 0, \\
 & a'^4 k'^4 + a'^5 k'^5 + a'^6 k'^6 + a'^7 k'^7 = 0, \\
 & b'^4 k'^4 + b'^5 k'^5 + b'^6 k'^6 + b'^7 k'^7 = 0, \\
 & c'^4 k'^4 + c'^5 k'^5 + c'^6 k'^6 + c'^7 k'^7 = 0, \\
 & a'^1 k'^5 + a'^2 k'^4 + a'^3 k'^3 + a'^4 k'^4 + a'^5 k'^5 = 0, \\
 & b'^1 k'^5 + b'^2 k'^4 + b'^3 k'^3 + b'^4 k'^4 + b'^5 k'^5 = 0, \\
 & c'^1 k'^5 + c'^2 k'^4 + c'^3 k'^3 + c'^4 k'^4 + c'^5 k'^5 = 0.
 \end{aligned}$$

Hence in general such a hyperquadric does not exist.

A necessary and sufficient condition for the existence of a non-singular quadric hypersurface V_1^1 with contact of the first order at each of the points P_i is that the rank of the matrix K of the coefficients of the equations (8.3) shall be 9.

$$K = \begin{vmatrix} a^{12} & a^{13} & a^{14} & a^{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ b^{12} & b^{13} & b^{14} & b^{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ c^{12} & c^{13} & c^{14} & c^{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ a^{21} & 0 & 0 & 0 & a^{23} & a^{24} & a^{25} & 0 & 0 & 0 \\ b^{21} & 0 & 0 & 0 & b^{23} & b^{24} & b^{25} & 0 & 0 & 0 \\ c^{21} & 0 & 0 & 0 & c^{23} & c^{24} & c^{25} & 0 & 0 & 0 \\ 0 & a^{31} & 0 & 0 & a^{32} & 0 & 0 & a^{34} & a^{35} & 0 \\ 0 & b^{31} & 0 & 0 & b^{32} & 0 & 0 & b^{34} & b^{35} & 0 \\ 0 & c^{31} & 0 & 0 & c^{32} & 0 & 0 & c^{34} & c^{35} & 0 \\ 0 & 0 & a^{41} & 0 & 0 & a^{42} & 0 & a^{43} & 0 & a^{45} \\ 0 & 0 & b^{41} & 0 & 0 & b^{42} & 0 & b^{43} & 0 & b^{45} \\ 0 & 0 & c^{41} & 0 & 0 & c^{42} & 0 & c^{43} & 0 & c^{45} \\ 0 & 0 & 0 & a^{51} & 0 & 0 & a^{52} & 0 & a^{53} & a^{54} \\ 0 & 0 & 0 & b^{51} & 0 & 0 & b^{52} & 0 & b^{53} & b^{54} \\ 0 & 0 & 0 & c^{51} & 0 & 0 & c^{52} & 0 & c^{53} & c^{54} \end{vmatrix}$$

There are ${}_{10}C_{10}$ square minors of K which must vanish. Hence we have ${}_{10}C_{10}$ conditions only six of which are independent. From these one may obtain the following ten conditions, six of which are independent

$$(8.4) \quad \begin{aligned} d^{13} d^{14} d^{15} - d^{23} d^{24} d^{25} &= 0, \\ d^{12} d^{15} d^{21} - d^{15} d^{22} d^{32} &= 0, \\ d^{14} d^{15} d^{21} - d^{15} d^{24} d^{44} &= 0, \\ d^{24} d^{45} d^{53} - d^{35} d^{52} d^{43} &= 0, \\ d^{12} d^{13} d^{21} - d^{13} d^{31} d^{21} &= 0, \\ d^{12} d^{24} d^{21} - d^{14} d^{41} d^{21} &= 0, \\ d^{12} d^{15} d^{51} - d^{15} d^{52} d^{21} &= 0, \\ d^{13} d^{24} d^{41} - d^{14} d^{43} d^{21} &= 0, \end{aligned}$$

$$d^{i's} d^{r'} d^{s'} - d^{i's} d^{s'} d^{r'} = 0,$$

$$d^{i's} d^{s'} d^{r'} - d^{i's} d^{r'} d^{s'} = 0.$$

Hence we state that the necessary and sufficient conditions for the existence of a non-singular contact hyperquadric of the quintuple of hypersurfaces at the points P_i are that the equations (8.4) are satisfied.

If we consider the hyperquadrics passing simply through the two points P_i , and P_j , ($i \neq j$), and having contact of the first order with the other three corresponding points P_h , P_k , P_l , ($h, k, l \neq i, j$; $h \neq k \neq l$), we find that these quadric hypersurfaces exist in each case

$$d^{hl} d^{lk} d^{kh} - d^{lh} d^{hk} d^{kl} \neq 0.$$

In general the above expression is not zero and hence such hyperquadrics exist. The conditions on the equation (8.2) for such a hyperquadric are

$$k^{mn} = 0 \quad (m, n \neq i, j).$$

In each case these hyperquadrics are made up of the two hyperplanes of the five-point of reference opposite P_i and P_j respectively. The equations are

$$y^i y^j = 0 \quad (i \neq j).$$

In general we may say that the only possibility of the existence of a hyperquadric passing simply through the points P_i and P_j , ($i \neq j$), and having contact of the first order at the remaining three corresponding points is that it consist of the hyperplanes $y^i = 0$, $y^j = 0$ each counted once.

There exists a unique hyperquadric passing simply through the four points P_j , ($j \neq 1$), and having contact of the second order at the point P_1 . These hyperquadrics are determined by the conditions

$$a^{i\beta} k^{i\beta} = 0, \quad b^{i\beta} k^{i\beta} = 0, \quad c^{i\beta} k^{i\beta} = 0,$$

$$[a^{ii} a^{i\beta} + \frac{1}{2}(a_{\check{\nu}}^{i\beta} + a^{i\alpha} a^{\alpha\beta})] k^{i\beta} + a^{i\alpha} a^{i\beta} k^{\alpha\beta} = 0,$$

$$[b^{ii} b^{i\beta} + \frac{1}{2}(b_{\check{\nu}}^{i\beta} + b^{i\alpha} b^{\alpha\beta})] k^{i\beta} + b^{i\alpha} b^{i\beta} k^{\alpha\beta} = 0,$$

$$[c^{ii} c^{i\beta} + \frac{1}{2}(c_{\check{\nu}}^{i\beta} + c^{i\alpha} c^{\alpha\beta})] k^{i\beta} + c^{i\alpha} c^{i\beta} k^{\alpha\beta} = 0,$$

$$[a^{i\beta} b^{ii} + a^{i\check{\nu}} b^{i\beta} + a_{\check{\nu}}^{i\beta} + a^{i\alpha} b^{\alpha\beta}] k^{i\beta} + (a^{i\alpha} b^{i\beta} + a^{i\beta} b^{i\alpha}) k^{\alpha\beta} = 0,$$

$$[a^{i\beta} c^{ii} + a^{i\check{\nu}} c^{i\beta} + a_{\check{\nu}}^{i\beta} + a^{i\alpha} c^{\alpha\beta}] k^{i\beta} + (a^{i\alpha} c^{i\beta} + a^{i\beta} c^{i\alpha}) k^{\alpha\beta} = 0,$$

$$[b^{i\beta} c^{ii} + b^{i\check{\nu}} c^{i\beta} + b_{\check{\nu}}^{i\beta} + b^{i\alpha} c^{\alpha\beta}] k^{i\beta} + (b^{i\alpha} c^{i\beta} + b^{i\beta} c^{i\alpha}) k^{\alpha\beta} = 0$$

$$(\alpha, \beta \neq 1; \alpha \neq \beta).$$

We shall next examine the situation for the existence of a hyperquadric having contact of the first order with the first associate quintuple. Since these associate varieties are tangent to the hyperplanes of the local five-point of reference it follows that the contact hyperquadric of this system is also tangent to the hyperplanes of the five-point of reference.

The equations of the hyperplanes of the five-point $P_1, \dots, P_5$ referred to the five-point $P_{y_1}, \dots, P_{y_5}$ are respectively

$$J^{\alpha i} y^{\alpha} = 0 \quad (1, \alpha = 1, \dots, 5; 1 \neq \alpha).$$

The equation of the most general hyperquadric passing through the five points P_{y_1} , referred to the local five-point

$P_{y_1}, \dots, P_{y_5}$ is

$$k^{\alpha\beta} y^\alpha y^\beta = 0 \quad (\alpha \neq \beta; \alpha < \beta).$$

This hyperquadric is tangent to the hyperplanes of the five-point P_i , ($i = 1, \dots, 5$), in case

$$\begin{aligned} k^{12}/\delta^{11} & \quad k^{13}/\delta^{11} & \quad k^{14}/\delta^{41} & \quad k^{15}/\delta^{51} & , \\ k^{12}/\delta^{12} & \quad k^{23}/\delta^{32} & \quad k^{24}/\delta^{42} & \quad k^{25}/\delta^{52} & , \\ k^{13}/\delta^{13} & \quad k^{23}/\delta^{23} & \quad k^{34}/\delta^{43} & \quad k^{35}/\delta^{53} & , \\ k^{14}/\delta^{14} & \quad k^{24}/\delta^{24} & \quad k^{34}/\delta^{34} & \quad k^{45}/\delta^{45} & , \\ k^{15}/\delta^{15} & \quad k^{25}/\delta^{25} & \quad k^{35}/\delta^{35} & \quad k^{45}/\delta^{45} & . \end{aligned}$$

Eliminating the k^{ij} we find as the necessary and sufficient conditions for the existence of a non-singular contact hyperquadric of the first associate quintuple at the corresponding points P_{q_i} the conditions

$$\begin{aligned} \delta^{23} \delta^{35} \delta^{42} - \delta^{24} \delta^{43} \delta^{31} &= 0, \\ \delta^{23} \delta^{35} \delta^{52} - \delta^{25} \delta^{53} \delta^{32} &= 0, \\ \delta^{24} \delta^{45} \delta^{52} - \delta^{25} \delta^{54} \delta^{42} &= 0, \\ \delta^{34} \delta^{45} \delta^{53} - \delta^{35} \delta^{54} \delta^{43} &= 0, \\ \delta^{12} \delta^{23} \delta^{31} - \delta^{21} \delta^{13} \delta^{32} &= 0, \\ \delta^{12} \delta^{24} \delta^{41} - \delta^{21} \delta^{14} \delta^{42} &= 0, \\ \delta^{13} \delta^{35} \delta^{51} - \delta^{11} \delta^{15} \delta^{53} &= 0, \\ \delta^{13} \delta^{34} \delta^{41} - \delta^{11} \delta^{14} \delta^{43} &= 0, \\ \delta^{14} \delta^{45} \delta^{51} - \delta^{11} \delta^{15} \delta^{54} &= 0, \end{aligned}$$

only six of which are independent.

The $\mathcal{S}^{ij}$ are formed from the d^{ij} by interchanging the superscripts of the a's, b's, and c's. Hence if we form the second associate quintuple we find that a non-singular contact hyperquadric of this system exists in case the equations (8.4) are satisfied. This may be generalized by the statement that a non-singular hyperquadric of the (k-2)-th associate quintuple exists if a non-singular contact hyperquadric of the k-th associate quintuple exists.

The existence of a quadric variety V_2^2 having first order contact at each point P_i restricts the fundamental system of varieties V_1 . When we examine the situation we find that the conditions (8.4) occur if, and only if, the points in the plane

$$y^i = 0, \quad y^j = 0,$$

where the lines l_{jk} , ($1, k \neq 1, j$; $1 \neq j$), of the five-point of reference cut the tangent hyperplanes of the respective varieties at x^m , ($m \neq 1, j, k, l$), in points which are collinear.

We shall denote the points in the plane $y^i = 0, y^j = 0$ where the line l_{jk} cuts the tangent hyperplane of the variety determined by x^m at the point P_m by P_m^{ij} . For example, the points P_2^{12} , P_3^{12} , P_4^{12} are respectively

$$(0, 0, d^{24}, -d^{23}, 0), \quad (0, d^{34}, 0, -d^{32}, 0), \quad (0, d^{43}, -d^{42}, 0, 0).$$

These points are collinear in case

$$d^{23} d^{34} d^{42} - d^{24} d^{43} d^{32} = 0.$$

Similarly we can obtain the other conditions of (8.4).

We may note here that these conditions are vacuously satisfied if all the d^{ij} are zero. It can readily be seen that any d^{ij} vanishes in case there exists on the variety which is the locus of x^i a curve such that the tangent to this curve cuts the variety which is the locus of x^j in the point P_j . If the determinant d of the d^{ij} vanishes the five corresponding hyperplanes coincide and the fundamental system is degenerate.

IX. CONCURRENCE CURVES AND COPLANE CURVES

A quintuple of corresponding curves on the five hypersurfaces will be called concurrence curves* in case the osculating hyperplanes at corresponding points are concurrent. Corresponding curves whose tangents at corresponding points meet the opposite hyperplanes of the local five-point of reference in five linearly dependent points will be called coplane curves.*

Any quintuple of curves, except curves on the parametric surface $u = \text{const.}$, may be defined by the equations $v = v(u)$, $w = w(u)$. The osculating hyperplanes of these curves will be determined by the points whose coordinates

are x^i , $\frac{dx^i}{du}$, $\frac{d^2x^i}{du^2}$, and $\frac{d^3x^i}{du^3}$, ($i = 1, \dots, 5$), where

$$\frac{dx^i}{du} = (a^{ir} + v^1 b^{ir} + w^1 c^{ir}) x^r = L^{ir} x^r,$$

$$\frac{d^2x^i}{du^2} (\alpha^{ir} + 2v^1 w^1 \gamma^{ir} + v^1 b^{ir} + w^1 c^{ir} + 2v^1 \lambda^{ir} + 2w^1 \mu^{ir} + v^1 \beta^{ir} + w^1 \gamma^{ir}) x^r = M^{ir} x^r,$$

*D. Sun, loc. cit.

$$\begin{aligned}
 \frac{d^3 x^i}{d u^3} = & (\alpha_u^{i\tau} a_u^{i\tau} a^{\alpha\tau} + a^{i\alpha} a^{\alpha\beta} a^{\beta\tau}) \\
 & + 3v^1 (\lambda_u^{i\tau} + a_u^{i\alpha} a^{\alpha\tau} + a^{i\alpha} a^{\alpha\beta} b^{\beta\tau}) \\
 & + 3w^1 (\mu_u^{i\tau} + a_u^{i\alpha} a^{\alpha\tau} + a^{i\alpha} a^{\alpha\beta} c^{\beta\tau}) \\
 & + 6v^1 w^1 (\lambda_w^{i\tau} + a_w^{i\alpha} c^{\alpha\tau} + a^{i\alpha} a^{\alpha\beta} b^{\beta\tau}) \\
 & + 3v^{12} (\lambda_v^{i\tau} + a_v^{i\alpha} b^{\alpha\tau} + a^{i\alpha} a^{\alpha\beta} b^{\beta\tau}) \\
 & + 3w^{12} (\mu_v^{i\tau} + a_v^{i\alpha} c^{\alpha\tau} + a^{i\alpha} a^{\alpha\beta} c^{\beta\tau}) \\
 & + v^{13} (\beta_v^{i\tau} + b_v^{i\alpha} b^{\alpha\tau} + b^{i\alpha} b^{\alpha\beta} b^{\beta\tau}) \\
 & + w^{13} (\gamma_v^{i\tau} + c_v^{i\alpha} c^{\alpha\tau} + c^{i\alpha} c^{\alpha\beta} c^{\beta\tau}) \\
 & + 3v^{12} w^1 (\beta_w^{i\tau} + b_w^{i\alpha} c^{\alpha\tau} + b^{i\alpha} b^{\alpha\beta} c^{\beta\tau}) \\
 & + 3v^1 w^{12} (\gamma_w^{i\tau} + b_w^{i\alpha} c^{\alpha\tau} + b^{i\alpha} c^{\alpha\beta} c^{\beta\tau}) \\
 & + 3\lambda^{i\tau} v^n + 3\mu^{i\tau} w^n + 2\beta^{i\tau} v^1 v^n + 2\gamma^{i\tau} w^1 w^n \\
 & + b^{i\tau} v^{1n} + c^{i\tau} w^{1n} \Big] x^\tau = N^{i\tau} x^\tau,
 \end{aligned}$$

and where

$$\begin{aligned}
 \alpha^{ij} &= a_u^{ij} + a^{i\alpha} a^{\alpha j}, \quad \beta^{ij} = b_v^{ij} + b^{i\alpha} b^{\alpha j}, \quad \gamma^{ij} = c_w^{ij} + c^{i\alpha} c^{\alpha j}, \\
 \lambda^{ij} &= a_v^{ij} + a^{i\alpha} b^{\alpha j}, \quad \mu^{ij} = a_w^{ij} + a^{i\alpha} c^{\alpha j}, \quad \nu^{ij} = b_w^{ij} + b^{i\alpha} c^{\alpha j}.
 \end{aligned}$$

Hence the osculating plane of the curve C at the point P has the equation

$$A^{12} y^2 + A^{13} y^3 + A^{14} y^4 + A^{15} y^5 = 0,$$

where the A^{ij} are the cofactors of the y^j , ($1 \neq j$) in the determinant

$$\begin{vmatrix}
 y^1 & y^2 & y^3 & y^4 & y^5 \\
 L^{12} & L^{13} & L^{14} & L^{15} & \\
 M^{12} & M^{13} & M^{14} & M^{15} & \\
 N^{12} & N^{13} & N^{14} & N^{15} &
 \end{vmatrix}$$

The equations of the four other osculating hyperplanes can be obtained by the cyclic substitution (1,2,3,4,5).

These osculating hyperplanes are concurrent in case

$$A = \begin{vmatrix} 0 & A^{12} & A^{13} & A^{14} & A^{15} \\ A^{21} & 0 & A^{23} & A^{24} & A^{25} \\ A^{31} & A^{32} & 0 & A^{34} & A^{35} \\ A^{41} & A^{42} & A^{43} & 0 & A^{45} \\ A^{51} & A^{52} & A^{53} & A^{54} & 0 \end{vmatrix} = 0,$$

which is a differential equation of the third order. Hence there exists a three parameter family of concurrence curves on each variety V_3 such that the corresponding hyperplanes of all the quintuples of curves are concurrent.

We have defined as coplane curves the curves C_i whose tangents at corresponding points meet the respective hyperplanes $y^i = 0$ in points which lie in the same hyperplane. The points are respectively

$$\begin{aligned} & (0, L^{12}, L^{13}, L^{14}, L^{15}), (L^{21}, 0, L^{23}, L^{24}, L^{25}) \\ & (L^{31}, L^{32}, 0, L^{34}, L^{35}), (L^{41}, L^{42}, L^{43}, 0, L^{45}) \\ & (L^{51}, L^{52}, L^{53}, L^{54}, 0). \end{aligned}$$

These points lie in the same hyperplane in case

$$I = \begin{vmatrix} 0 & L^{12} & L^{13} & L^{14} & L^{15} \\ L^{21} & 0 & L^{23} & L^{24} & L^{25} \\ L^{31} & L^{32} & 0 & L^{34} & L^{35} \\ L^{41} & L^{42} & L^{43} & 0 & L^{45} \\ L^{51} & L^{52} & L^{53} & L^{54} & 0 \end{vmatrix} = 0$$

which is an equation of the first order and of degree five. Hence through each point on each variety V_3 of the fundamental system there pass five coplane curves.

X. PARAMETRIC SURFACES AND LAPLACE TRANSFORMS

Each variety V_3 of the quintuple can be considered as referred to three one-parameter families of parametric surfaces $u = \text{const.}$, $v = \text{const.}$, and $w = \text{const.}$ Each surface of one family is cut by the surfaces of each of the other families in a one-parameter family of curves. The intersection of the surfaces $v = \text{const.}$ and $w = \text{const.}$ will be called u -curves and denoted by C_u . Similarly we have the curves C_v and C_w . It is known that every surface in a space S_4 sustains a conjugate net and hence that there exist second order partial differential equations in two independent variables whose fundamental system of solutions will be the projective homogeneous coordinates of these parametric surfaces. For example, the coordinates $x = x(u, v, a)$ of the family of surfaces $w = a = \text{const.}$ will satisfy a second order partial differential equations in the independent variables u and v . In this section we shall show how these equations can be obtained from the fundamental system of equations. Since every surface in a space S_4 sustains a conjugate net it is possible to find the Laplace equation of the net of curves in which the surfaces of any two families cut those of the third family. Laplace transforms of these nets are known. The case where it is possible to refer the three one-parameter families of surfaces simultaneously to these conjugate nets will be discussed. The varieties one-parameter family of whose parametric surfaces are the Laplace

transforms of a family of surfaces on the original variety will be noted.

We can eliminate the x^j , ($j \neq 1$), from the nine equations

$$\begin{aligned} x_u^i &= a^{i\alpha} x^\alpha, & x_v^i &= b^{i\alpha} x^\alpha, & x_w^i &= c^{i\alpha} x^\alpha, \\ x_{uu}^i &= \alpha^{i\alpha} x^\alpha, & x_{vv}^i &= \beta^{i\alpha} x^\alpha, & x_{ww}^i &= \gamma^{i\alpha} x^\alpha, \\ x_{uw}^i &= \lambda^{i\alpha} x^\alpha, & x_{vw}^i &= \mu^{i\alpha} x^\alpha, & x_{vw}^i &= \nu^{i\alpha} x^\alpha, \end{aligned}$$

so as to obtain the differential equations of the nets of curves in which the surfaces of the families $w = \text{const.}$, $v = \text{const.}$, and $u = \text{const.}$ are cut by the other two families. For example, the surfaces $w = \text{const.}$ are given by

$$\begin{aligned} x_u^i &= a^{i\alpha} x^\alpha, & x_v^i &= b^{i\alpha} x^\alpha, \\ x_{uu}^i &= \alpha^{i\alpha} x^\alpha, & x_{vv}^i &= \beta^{i\alpha} x^\alpha, & x_{uv}^i &= \lambda^{i\alpha} x^\alpha. \end{aligned}$$

Eliminating x^j , ($j \neq 1$), from these five equations we have

$$(10.1) \quad A_i x_{uu}^i + B_i x_{uv}^i + C_i x_{vv}^i + D_i x_u^i + E_i x_v^i + F_i x^i = 0.$$

Similarly for the surfaces $v = \text{const.}$ and $u = \text{const.}$ we obtain respectively

$$\begin{aligned} A_i x_{vv}^i + B_i x_{vw}^i + C_i x_{ww}^i + D_i x_v^i + E_i x_w^i + F_i x^i &= 0, \\ A_i x_{ww}^i + B_i x_{wu}^i + C_i x_{uu}^i + D_i x_w^i + E_i x_u^i + F_i x^i &= 0. \end{aligned}$$

where

$$\begin{aligned} A_i &= (\beta, \lambda, a, b), & A_i &= (\gamma, \nu, b, c), & A_i &= (\alpha, \mu, a, c), \\ B_i &= (\alpha, \beta, a, b), & B_i &= (\beta, \gamma, b, c), & B_i &= (\alpha, \gamma, a, c), \\ C_i &= -(\alpha, \lambda, a, b), & C_i &= -(\beta, \nu, b, c), & C_i &= -(\gamma, \mu, a, c), \\ D_i &= -(\alpha, \beta, \lambda, b), & D_i &= -(\beta, \gamma, \nu, c), & D_i &= -(\alpha, \gamma, \mu, a), \end{aligned}$$

$$E_1^i = (\alpha, \beta, \lambda, a), \quad E_2^i = (\beta, \gamma, \nu, b), \quad E_3^i = (\alpha, \gamma, \mu, c),$$

$$F_1^i = (\alpha, \beta, \lambda, a, b), \quad F_2^i = (\beta, \gamma, \nu, b, c), \quad F_3^i = (\alpha, \gamma, \mu, a, c),$$

in which the fourth order determinants (a, b, c, d) have diagonal elements a^{ij} , b^{ih} , c^{il} , d^{ik} , and the fifth order determinants (a, b, c, d, e) have diagonal elements a^{ij} , b^{ih} , c^{il} , d^{ik} , e^{ik} , $(j < h < l < k; i \neq j, h, l, k)$.

By the change of the independent variable

$$(2.5) \quad \bar{u} = \phi(u, v, w), \quad \bar{v} = \psi(u, v, w), \quad \bar{w} = \theta(u, v, w)$$

the hypersurfaces of the quintuple are left unchanged, but the parametric surfaces $u = \text{const.}$, $v = \text{const.}$, $w = \text{const.}$ are transformed into the surfaces $\bar{u} = \text{const.}$, $\bar{v} = \text{const.}$, $\bar{w} = \text{const.}$, and the curves C_u , C_v , C_w , of intersection into $C_{\bar{u}}$, $C_{\bar{v}}$, $C_{\bar{w}}$, respectively

In each case we shall hold fixed $\bar{w}$, $\bar{v}$, and $\bar{u}$ and apply the resulting transformation to the equations of the respective families of surfaces. For example, when $\bar{w} = \text{const.}$ we can solve the resulting equation $\theta(u, v, w) = \text{const.}$ for w in terms of u and v and substitute this value in the other two equations of transformation. We have then

$$\bar{u} = \phi(u, v, w(u, v)), \quad \bar{v} = \psi(u, v, w(u, v)), \quad \bar{w} = \text{const.}$$

When we apply this transformation to equation (10.1) we have the first equation of (10.3). Similarly we may set $\bar{v} = \text{const.}$, and $\bar{u} = \text{const.}$ In this way we obtain as the differential equations of the curves on the new parametric surfaces

$$\begin{aligned}
 & \bar{A}_1 \bar{x}_{uu} + \bar{B}_1 \bar{x}_{uv} + \bar{C}_1 \bar{x}_{vv} + \bar{D}_1 \bar{x}_u + \bar{E}_1 \bar{x}_v + \bar{F}_1 \bar{x} = 0, \\
 (10.3) \quad & \bar{A}_2 \bar{x}_{vv} + \bar{B}_2 \bar{x}_{vw} + \bar{C}_2 \bar{x}_{ww} + \bar{D}_2 \bar{x}_v + \bar{E}_2 \bar{x}_w + \bar{F}_2 \bar{x} = 0, \\
 & \bar{A}_3 \bar{x}_{ww} + \bar{B}_3 \bar{x}_{wu} + \bar{C}_3 \bar{x}_{uu} + \bar{D}_3 \bar{x}_w + \bar{E}_3 \bar{x}_v + \bar{F}_3 \bar{x} = 0,
 \end{aligned}$$

where the subscripts 1 indicating the particular variety which is being studied have been dropped, and where

$$\begin{aligned}
 \bar{A}_1 &= A_1 (\phi_u + \phi_w w_u)^2 + B_1 (\phi_u + \phi_w w_u)(\phi_v + \phi_w w_v) \\
 &\quad + C_1 (\phi_v + \phi_w w_v), \\
 \bar{B}_1 &= 2A_1 (\phi_u + \phi_w w_u)(\psi_u + \psi_w w_u) \\
 &\quad + B_1 [(\phi_u + \phi_w w_u)(\psi_v + \psi_w w_v) + (\phi_v + \phi_w w_v)(\psi_u + \psi_w w_u)], \\
 &\quad + 2C_1 (\phi_v + \phi_w w_v)(\psi_v + \psi_w w_v), \\
 \bar{C}_1 &= A_1 (\psi_u + \psi_w w_u)^2 + B_1 (\psi_u + \psi_w w_u)(\psi_v + \psi_w w_v) \\
 &\quad + C_1 (\psi_v + \psi_w w_v).
 \end{aligned}$$

A , B , C , and $\bar{A}$, $\bar{B}$, $\bar{C}$ can be obtained from the above by the substitution

$$\begin{pmatrix} 1 & \phi & \psi & u & w \\ 2 & \psi & \theta & v & u \\ 3 & \theta & \psi & w & v \end{pmatrix}$$

From this it follows that if ϕ and ψ ; ψ and θ ; and θ and ϕ , are respectively solutions of the equations

$$\begin{aligned}
 & A_1 (x_u + x_w w_u)^2 + B_1 (x_u + x_w w_u)(x_v + x_w w_v) + C_1 (x_v + x_w w_v)^2 = 0, \\
 (10.4) \quad & A_2 (x_v + x_w w_v) + B_1 (x_v + x_w w_v)(x_u + x_w w_u) + C_2 (x_u + x_w w_u) = 0, \\
 & A_3 (x_w + x_v w_v)^2 + B_3 (x_w + x_v w_v)(x_u + x_v w_u) + C_3 (x_u + x_v w_u)^2 = 0,
 \end{aligned}$$

the members of the system (10.3) are of the Laplace type. If the systems of two linear first order partial differential

obtained by factoring the quadratic equations (10.4) are complete* it is possible to find a transformation of the type (2.5) such that the system of equations (10.3) are of the Laplace type. For example the system of equations

$$A(\phi) = (\phi_u + \phi_w w_u) + \lambda (\phi_v + \phi_w w_v) = 0,$$

$$B(\phi) = (\phi_u - \phi_v v_w) + \mu (\phi_u + \phi_v v_u) = 0,$$

form a complete system determining ϕ , in case the commutator

$$A(B(\phi)) - B(A(\phi))$$

is a linear combination of $A(\phi)$ and $B(\phi)$. If the systems are not complete they may be made so by adjoining to each a third equation of the same type, namely the commutator set equal to zero. In this case the solutions are $\phi = \text{const.}$, $\psi = \text{const.}$, and $\theta = \text{const.}$, respectively.

If $B^2 - 4AC = 0$, we have instead of the net of curves two coincident families of curves which are the asymptotic curves of the surfaces

As a result of this transformation the nets of curves in which two families of parametric surfaces cut the third are now represented by a system of second order partial differential equations of Laplace,

*Goursat-Hedrick, loc. cit.

$$\begin{aligned}x_{uv} &= a_1 x_u + b_1 x_v + d_1 x, \\x_{vw} &= b_2 x_v + c_2 x_w + d_2 x, \\x_{wu} &= a_3 x_u + c_3 x_w + d_3 x,\end{aligned}$$

and hence the parametric curves on each surface $u = \text{const.}$, $v = \text{const.}$, and $w = \text{const.}$ are the curves of the conjugate net on that surface.

The transformation of the dependent variable

$$x = \lambda(u, v, w) \bar{x}$$

leaves the parametric curves and surfaces unaltered, but changes the coefficients of the system of equations. Upon applying this transformation to (10.5) we arrive at a new system of the same type where we shall denote the new coefficients by a , b , c , and d , where

$$\begin{aligned}\bar{a}_1 &= -\frac{\lambda_v}{\lambda} + a_1, & \bar{b}_1 &= -\frac{\lambda_u}{\lambda} + b_1, \\ \bar{d}_1 &= -\frac{\lambda_{uv}}{\lambda} + a_1 \frac{\lambda_w}{\lambda} + b_1 \frac{\lambda_v}{\lambda} + d_1, \\ \bar{b}_2 &= -\frac{\lambda_w}{\lambda} + b_2, & \bar{c}_2 &= -\frac{\lambda_v}{\lambda} + c_2, \\ \bar{d}_2 &= -\frac{\lambda_{vw}}{\lambda} + b_2 \frac{\lambda_u}{\lambda} + c_2 \frac{\lambda_w}{\lambda} + d_2, \\ \bar{c}_3 &= -\frac{\lambda_u}{\lambda} + c_3, & \bar{a}_3 &= -\frac{\lambda_w}{\lambda} + a_3, \\ \bar{d}_3 &= -\frac{\lambda_{wu}}{\lambda} + c_3 \frac{\lambda_v}{\lambda} + a_3 \frac{\lambda_w}{\lambda} + d_3.\end{aligned}$$

It is possible to determine λ so that $\bar{d}_1 = \bar{d}_2 = \bar{d}_3 = 0$. For simplicity in calculations the resulting canonical form will be used in the following pages,

$$\begin{aligned}(10.8) \quad x_{uv} &= a_1 x_u + b_1 x_v, \\ x_{vw} &= b_2 x_v + c_2 x_w, \\ x_{wu} &= a_3 x_u + c_3 x_w.\end{aligned}$$

Any transformation of the independent variables of the type

$$(10.9) \quad \bar{u} = U(u), \quad \bar{v} = V(v), \quad \bar{w} = W(w),$$

leaves the hypersurface together with the parametric curves and surfaces unchanged, and preserves the form of the canonical system (10.8). Under this transformation the new coefficients are

$$\begin{aligned} \bar{a}_1 &= \frac{1}{V'} a_1, & \bar{b}_1 &= \frac{1}{U'} b_1, \\ \bar{b}_2 &= \frac{1}{W'} b_2, & \bar{c}_2 &= \frac{1}{V'} c_2, \\ \bar{c}_3 &= \frac{1}{U'} c_3, & \bar{a}_3 &= \frac{1}{W'} a_3, \end{aligned}$$

We shall next obtain the equations of the point transformations to the varieties the nets on one of whose parametric surfaces shall be the Laplace transforms of the nets of one family of parametric surfaces of the given variety. We shall call these varieties the Laplace transforms of the given variety. They give the point correspondences between the three focal varieties of a triply infinite system C_3 of lines in the space S_4 . Through every line of the system pass three developables. That is, every system C_3 of lines in S_4 can be grouped into ∞^2 developables in three ways.

1. P. Eisenhart, Transformations of Surfaces, p. 17.

1. P. Eisenhart, Differential Geometry, pp. 403 ff.

C. Segre, Un' Osservazione sur Sistemi di Rette degli Spazi Superiori, Rendiconti del Circolo Matematico di Palermo, II, (1888), pp. 148-149.

Each variety V_j has six Laplace transforms, which we shall designate as the u, v, and w transforms. They are

$$\begin{aligned}x' &= x - 1/b \, x_u, & x' &= x - 1/c \, x_u, \\x'' &= x - 1/a \, x_v, & x'' &= x - 1/c \, x_v, \\x''' &= x - 1/a \, x_w, & x''' &= x - 1/b \, x_w.\end{aligned}$$

We can readily show that the developables of the system C_j touch the focal varieties along the curves of the conjugate webs. That is, the point coordinates satisfy systems of equations of the type (10.8).

We may note that if $b_1 = c_3$ the two u-transforms coincide. Similarly, if $a_1 = c_2$ and $a_3 = b_2$ the v-transforms and w-transforms coincide. If $b_1 = c_3$ and $\frac{\partial}{\partial v} 1/b_1 = a_3/b_1$, and $\frac{\partial}{\partial w} 1/b_1 = a_3/b_1$ the u-transforms coincide and degenerate into curves: If $\frac{\partial}{\partial v} 1/b_1 = a_1/b_1$ and $\frac{\partial}{\partial w} 1/c_3 = a_3/c_3$ the u-transforms degenerate into surfaces.

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VITA

Beatrice Liberty Hagen was born near Ellinwood, Kansas, July 4, 1899. She received her elementary and secondary education in the public schools of that district. She was granted an A.B. degree from the University of Kansas in 1920 and an A.M. degree from the University of Chicago in 1926. At Chicago she studied under Professors Barnard, Bliss, Dickson, Graves, Lane, Logsdon, Moore, Moulton, and Slaught.

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